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Non-linear electrodynamics in curved backgrounds

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Abstract: We study non-linear electrodynamics in curved space from the viewpoint of dualities. After establishing the existence of a topological bound for self-dual configurations of Born-Infeld field in curved space, we check that the energy-momentum tensor vanishes. These properties are shown to hold for general duality-invariant non-linear electrodynamics. We give the dimensional reduction of Born-Infeld action to three dimensions in a general curved background admitting a Killing vector. The SO(2) duality symmetry becomes manifest but other symmetries present in flat space are broken, as is U-duality when one couples to gravity. We generalize our arguments on duality to the case of n U(1) gauge fields, and present a new lagrangian possessing $\mathrm{SO}(n) \times \mathrm{SO}(2)_{\mathrm{elemag}}$ duality symmetry. Other properties of this model such as Legendre duality and enhancement of the symmetry by adding dilaton and axion, are studied. We extend our arguments to include a background b-field in the curved space, and give new examples including almost Kähler manifolds and Schwarzshild black holes with a b-field.

Keywords: Duality in Gauge Field Theories, D-branes, Classical Theories of Gravity, Non-Commutative Geometry.

Contents

1. Introduction

Non-linear electrodynamics has been studied for a long time. M. Born and L. Infeld [1] introduced a version of it which has received renewed attention since it has turned out to play an important role in the development in string theory. The particular non-linear electrodynamics of the Born-Infeld lagrangian describes low energy physics on D-branes which are intrinsically non-perturbative solitonic objects in string theory. The introduction of D-brane has expanded the possibilities for constructing realistic models in string theory, and one of those possibilities is the so-called "brane-world" scenario which assumes that our spacetime is a worldvolume of the (D-)branes. Therefore this scenario naturally introduces non-linear electrodynamics into gauge theories.

Since string theory contains gravity as its fundamental excitation, it is plausible that the low energy effective action on the D-brane incorporates gravity. However, the effective action for D-brane was derived initially only in a flat background. Therefore the precise form of non-linear electrodynamics on the D-brane in arbitrary gravity background, while probably given by the covariant form of the Born-Infeld lagrangian [2], is still not yet known with certainty, especially in the case of more than one U(1) gauge field. Recently, in the supergravity context, progress on the construction of D-brane solutions possessing non-flat worldvolumes has been reported [3, 4, 5]. These are called Ricci-flat branes. If we accept the brane-world scenario, then surely we have curved spacetime as our universe, thus we need to understand better the relation between non-linear electrodynamics and gravity .

In this paper, we shall study classical non-linear electrodynamics in curved background. Since string theory is believed to be controlled by (non-)perturbative duality symmetry and in particular S-duality may be realized as a *non-linear Legendre (electric-magnetic) duality* on the D3-brane worldvolume theory, we concentrate on the duality properties of the non-linear electrodynamics in curved backgrounds. In particular, self-dual configurations of the gauge field are closely related to Legendre duality, therefore we study mainly the "instanton" configurations. In the string theory context, self-dual (or anti-self-dual) configurations are concerned with stable (BPS) brane configurations such as bound states of branes.

We see that the self-dual configurations in the curved gravity background are involved with various physical requirements which constrain the explicit form of the lagrangian of the non-linear electrodynamics. Our aim is to find lagrangian(s) which possesses desirable properties concerning the self-duality and self-dual configurations.

Another motivation of this paper concerns non-commutative geometry. String theory in a background NS-NS two-form b-field gives rise to exotic worldvolume physics on the D-brane, involving "non-commutative" spacetimes. The non-commutativity of the worldvolume can be introduced through a $*$-product in the multiplication of functions on that space. This exotic theory is, as shown in ref. [6], equivalent with the ordinary Born-Infeld type non-linear electrodynamics in a background b-field. The equivalence involves non-linear electrodynamics in an essential way, one must go beyond Maxwell's linear electrodynamics.

From this point of view, we introduce the b-field in non-linear electrodynamics on a curved background. Consequently the b-field is no longer strictly constant. It may be covariantly constant, it may have constant magnitude or it may merely tend to a constant field near infinity. We shall call such manifolds with a non-trivial b-field "non-commutative manifolds". The analysis of the self-dual configurations of the gauge fields is extended to include the case of non-commutative manifolds in this sense.

The organization of this paper is as follows. First in the next section we shall consider the abelian Born-Infeld theory in the curved background, and see that the notion of the topological bound for instantons survives in the case of curved gravity background. In section 3, we extend the analysis to general non-linear electrodynamics, and show that the properties concerning the instantons can be derived from the duality invariance condition of the system. Section 4 is devoted to the study of the case of several gauge fields. There are many versions of extension of the abelian Born-Infeld theory, however we propose a new lagrangian which possesses intriguing duality properties. In section 5, subsequently we study various properties of non-linear electrodynamics such as hamiltonian formalism, Legendre duality, symmetry enhancement by adding dilaton and axion, and dimensional reduction in curved space. Section 6 includes many examples of non-commutative manifolds and self-dual configuration on those manifolds. We construct a non-commutative generalization of the Schwarzshild black hole.

2. Born-Infeld system in curved background

First let us study Born-Infeld theory, the famous non-linear extension of the Maxwell theory. This theory exhibits many notable properties such as duality invariance. This property becomes a guiding principle for constructing other non-linear extensions of non-linear electrodynamics [7]. Some general argument for the form of non-linear electrodynamics in curved background will be presented in later sections.

2.1 Self-duality and topological bound

In this subsection we shall give a topological bound for euclidean Born-Infeld action on a riemannian manifold and show that it is attained only by the (anti-)self-dual gauge fields. If $(\mathcal{M}, g_{\mu\nu})$ is a riemannian 4-manifold, then the action of the Born-Infeld system in this background manifold is

$$I(F) = \int_{\mathcal{M}} d^4x \left(\sqrt{\det\left(g_{\mu\nu} + \mathcal{F}_{\mu\nu}\right)} - \sqrt{g} \right), \tag{2.1}$$

where we have defined $g \equiv \det g_{\mu\nu}$, and

$$\mathcal{F}_{\mu\nu} \equiv F_{\mu\nu} + b_{\mu\nu}\,. \tag{2.2}$$

Here b is a background two-form which is not necessarily constant, but we assume that

$$d\mathcal{F} = 0\,. \tag{2.3}$$

Noting that from the (anti-)symmetric property of the indices of the tensors, we have $\det(g_{\mu\nu} + \mathcal{F}_{\mu\nu}) = \det(g_{\mu\nu} - \mathcal{F}_{\mu\nu})$, thus

$$\left[\det\left(g_{\mu\nu} + \mathcal{F}_{\mu\nu}\right)\right]^2 = \det\left(g_{\mu\nu} + \mathcal{F}_{\mu\rho}\mathcal{F}_{\nu}{}^{\rho}\right) g\,. \tag{2.4}$$

Using Minkowski's inequality[1]

$$\left[\det\left(g_{\mu\nu} + \mathcal{F}_{\mu\rho}\mathcal{F}_{\nu}{}^{\rho}\right)\right]^{1/4} \geq \left[\det\left(g_{\mu\nu}\right)\right]^{1/4} + \left[\det\left(\mathcal{F}_{\mu\rho}\mathcal{F}_{\nu}{}^{\rho}\right)\right]^{1/4}, \tag{2.5}$$

we obtain a bound for the action (2.1) as

$$I(F) \geq \left| \int_{\mathcal{M}} d^4x \sqrt{\det \mathcal{F}_{\mu\nu}} \right|. \tag{2.6}$$

The quantity appearing in the right hand side is a topological invariant since

$$\sqrt{\det \mathcal{F}_{\mu\nu}} = \frac{1}{4} \left| \sqrt{g} \mathcal{F}_{\mu\nu} * \mathcal{F}^{\mu\nu} \right|, \tag{2.7}$$

where we have defined the Hodge dual of the field strength as

$$*\mathcal{F}^{\mu\nu} \equiv \frac{1}{2} \eta^{\mu\nu\rho\sigma} \mathcal{F}_{\rho\sigma}\,. \tag{2.8}$$

Here η is a covariant anti-symmetric tensor. The equality in (2.6) holds if $g_{\mu\nu} \propto \mathcal{F}_{\mu\rho}\mathcal{F}_{\nu}{}^{\rho}$, and it is easy to show that this relation results in the (anti-)self-duality condition

$$\mathcal{F}_{\mu\nu} = \pm * \mathcal{F}_{\mu\nu}\,. \tag{2.9}$$

From the above argument we observe that in a general curved background metric the Born-Infeld action is bounded by a topological quantity, and the bound is realized when the gauge field configuration is (anti-)self-dual.

2.2 The energy momentum tensor

The above argument supposes a given background metric, however if one considers a combined system of gravity and a gauge field then the gauge field will, in general,

[1] This inequality can be easily verified in a local orthonormal frame.

affect the metric. Here we shall show that in the case of Born-Infeld theory self-dual configuration (which we have shown to give a topological bound for the action) does not affect the background.

By differentiating the action (2.1) with respect to the metric, it is easy to obtain the energy momentum tensor for this system:

$$T^{\mu\nu} = \frac{1}{4}\sqrt{\det\left(g_{\mu\nu}+\mathcal{F}_{\mu\nu}\right)}\left(\left(g_{\mu\nu}+\mathcal{F}_{\mu\nu}\right)^{-1}+\left(g_{\mu\nu}-\mathcal{F}_{\mu\nu}\right)^{-1}\right)-\frac{1}{2}\sqrt{\det g_{\mu\nu}}g^{\mu\nu}\,. \quad (2.10)$$

Using the technique used in the previous subsection, this expression can be cast into the form

$$\begin{aligned} T^{\mu\nu} &= \frac{1}{2}\left[\det\left(g_{\mu\nu}+\mathcal{F}_{\mu\rho}\mathcal{F}_{\nu}{}^{\rho}\right)\right]^{1/4}\left(\det g_{\mu\nu}\right)^{1/4}\left(g_{\mu\nu}+\mathcal{F}_{\mu\rho}\mathcal{F}_{\nu}{}^{\rho}\right)^{-1}- \\ &\quad -\frac{1}{2}\sqrt{\det g_{\mu\nu}}g^{\mu\nu}\,. \end{aligned} \quad (2.11)$$

Now, for the self-dual configuration which saturates Minkowski's inequality, the gauge field configuration satisfies[2]

$$\mathcal{F}_{\mu\rho}\mathcal{F}_{\nu}{}^{\rho} = \frac{1}{4}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}g_{\mu\nu}\,. \quad (2.12)$$

Substituting this condition into the expression for the energy momentum tensor above, we obtain

$$T^{\mu\nu} = 0\,. \quad (2.13)$$

This shows that in the combined system

$$I = I(F) + \int_{\mathcal{M}} d^4x\sqrt{g}R(g) \quad (2.14)$$

this action is extremized on an Einstein manifold with an (anti-)self-dual gauge field configuration. The Einstein equation is not modified by the presence of the matter gauge field. The question of whether we have a true lower bound is slightly delicate. Because the Einstein action is not bounded below, the solution is only an extremum, not a minimum. However with respect to variations of the gauge field, keeping the metric and the topological term fixed, it is an absolute minimum. If we had in mind the idea that the background metric was a Kähler metric, then in fact the Einstein action when restricted to Kähler metrics is purely topological [8], once the volume has been fixed.

The fact that any instanton configuration does not affect the background metric can be seen also in the context of string theory. In equations of motion of supergravity, under a particular ansatz of the dilaton and R-R scalar field concerning self-dual point, the energy momentum tensor of them vanishes [9, 10, 11]. Thus this D-instantons does not affect the Einstein equation. Inclusion of the b-field was discussed in ref. [12].

[2]This equality can be verified easily in local othonormal frame.

2.3 Open and closed string metric

Let us comment on the meaning of the background b-field. Since we are considering curved backgrounds, this b-field is expected to be non constant, under some physical circumstance such as the backgrounds preserving some of the supersymmetries. So in the following study, the b-field is not necessarily a constant.

Following the argument of ref. [6], we can define the "open string metric" G as

$$G_{\mu\nu} \equiv g_{\mu\nu} - B_{\mu\rho} g^{\rho\sigma} B_{\sigma\nu} \,. \tag{2.15}$$

This is a simple generalization of the one given in ref. [6] in which only the constant metric and the constant b-field were considered. Note that the additional term in eq. (2.15) contributes positively so that both g and G are positive definite. A simple choice of the background b-field is one which is (anti-)self-dual with respect to the "closed string metric" $g_{\mu\nu}$. Then from similar argument to those presented around eq. (2.12), one can see that $g^{-1}Bg^{-1}B$ is proportional to the identity matrix. Using the definition (2.15), we obtain an interesting result: the open string metric is conformally equivalent to the closed string metric,

$$g_{\mu\nu} \propto G_{\mu\nu} \,. \tag{2.16}$$

As is well known, the notion of (anti-)self-duality is conformally invariant:

$$F_{\mu\nu} = \pm \frac{1}{2} g_{\mu\rho} g_{\nu\sigma} \eta^{\rho\sigma\tau\lambda} F_{\tau\lambda} \tag{2.17}$$

is invariant under a Weyl rescaling of the metric

$$g_{\mu\nu} \longrightarrow \Omega^2 g_{\mu\nu} \,, \qquad \eta^{\mu\nu\rho\sigma} \longrightarrow \Omega^{-4} \eta^{\mu\nu\rho\sigma} \,. \tag{2.18}$$

So we are lead to the following fact: if b is (anti-)self-dual with respect to the closed string metric g, then the (anti-)self-duality of b holds also with respect to the open string metric G.

It is easy to see that the inverse of the above claim holds. What one has to note is that there is a inverse solution of eq. (2.15) in the matrix notation

$$g = G + bg^{-1}b = G\left(1 + G^{-1}b\, G^{-1}b + \cdots\right) , \tag{2.19}$$

where the ellipsis indicates an infinite power series in $G^{-1}b\, G^{-1}b$. So if b is (anti-)self-dual with respect to G, the matrix $G^{-1}b\, G^{-1}b$ is in proportion to the identity matrix, then we see that $g \propto G$ holds. Thus

$$b = \pm *_g\, b \Longleftrightarrow b = \pm *_G\, b \,. \tag{2.20}$$

In conclusion, as long as the b-field is (anti-)self-dual with respect to closed or open string metric, both agree about the (anti-)self-duality of the gauge field configuration.

3. General non-linear electrodynamics

We now consider a more general type of the non-linear electrodynamics:

$$I(F) = \int d^4x \, \sqrt{g} L_F \,. \tag{3.1}$$

Since we intend to study self-dual configuration in a curved background, the appropriate property required on the lagrangian[3] L_F is *duality invariance.* The duality invariance of general non-linear electrodynamics was studied in detail in refs. [7, 15, 16, 17], in which the authors adopted the following form of the duality rotation:

$$\begin{cases} \delta F_{\mu\nu} = *P_{\mu\nu} \\ \delta P_{\mu\nu} = *F_{\mu\nu} \,, \end{cases} \tag{3.2}$$

where the tensor P is defined[4] as

$$P^{\mu\nu} \equiv 2 \frac{\partial L_F}{\partial F_{\mu\nu}} \,. \tag{3.3}$$

Note that the duality transformation transforms F into $*P$, not into $*F$. This is a precise generalization of the electro-magnetic duality of the Maxwell theory. The duality of this theory rotates $\mathbf{E}$ and $\mathbf{H}$ (or $\mathbf{D}$ and $\mathbf{B}$). Of course if we adopt the Maxwell lagrangian in vacuum, $L_F = F_{\mu\nu}F^{\mu\nu}/4$, then we have $P = F$.

3.1 Conditions concerning the self-dual configurations

Our aim is to clarify the constraints on the form of the lagrangian L_F by requiring some physical properties concerning the self-dual configurations. The first requirement is that the well-known (anti-)self-duality condition

$$F = \epsilon * F \tag{3.4}$$

with $\epsilon = \pm 1$ must be reduced to the natural (anti-)self-duality condition expected from the above rotation (3.2):

$$F = \epsilon * P \,. \tag{3.5}$$

In fact, we see that any non-trivial solution of the condition (3.5) should also be the solution of (3.4) and this restricts the form of the lagrangian. Let us express the explicit form of the lagrangian as

$$L_F = L_F(X, Y) \,, \tag{3.6}$$

[3] Since we are in the euclidean regime we use the word lagrangian to denote the action density. It is the negative of the analytic continuation of the usual real time lagrangian.

[4] We change G (notation in the previous literature) to P because nowadays the former is usually taken to denote the open string metric.

where we have defined Lorentz-invariant quantities $X \equiv F_{\mu\nu}F^{\mu\nu}/4, Y \equiv F_{\mu\nu} * F^{\mu\nu}/4$. Then from this expression we obtain

$$P^{\mu\nu} = \left(F^{\mu\nu}\frac{\partial L_F}{\partial X} + *F^{\mu\nu}\frac{\partial L_F}{\partial Y} \right) . \tag{3.7}$$

Substituting the natural (anti-)self-duality (3.5) into the definition above, we have the equation

$$\left(\epsilon - \frac{\partial L_F}{\partial Y} \right) * F = \left(\frac{\partial L_F}{\partial X} \right) F . \tag{3.8}$$

Assuming that this equation has non-trivial ($F \neq 0$) solution, it must also be a solutions of the usual self-duality equation (3.4), and so we have a condition on the lagrangian

$$\left[1 - \frac{\partial L_F}{\partial X} - \epsilon\frac{\partial L_F}{\partial Y} \right]_{X=\epsilon Y} = 0 . \tag{3.9}$$

Here we have used the fact the self-duality condition implies $X = \epsilon Y$. In the case of Maxwell lagrangian $L_F = X$, the constraint above is trivially satisfied.

A second physical requirement on the lagrangian is that the self-dual solution must be a solution of second order equations of motion. The equation of motion for this non-linear system is $\nabla_\nu P^{\mu\nu} = 0$. Substituting eq. (3.7) into this equation, then we have

$$\left[\frac{\partial^2 L_F}{\partial X^2} + 2\epsilon\frac{\partial^2 L_F}{\partial X \partial Y} + \frac{\partial^2 L_F}{\partial Y^2} \right]_{X=\epsilon Y} = 0 . \tag{3.10}$$

When obtaining this equation, we have used the equation $\nabla_\mu F^{\mu\nu} = 0$ which comes from the Bianchi identity and the (anti-)self-duality condition (3.4).

The final requirement is that as seen in the previous section the background adopted must not be affected by the (anti-)self-dual configuration considered. Let us calculate the energy momentum tensor. Using an expression given in ref. [13],

$$T_{\mu\nu} = \frac{\sqrt{g}}{2} \left(g_{\mu\nu}L_F - P_\mu{}^\lambda F_{\nu\lambda} \right) . \tag{3.11}$$

Substituting the explicit expression of P (3.7) and noting the relation $F_\mu{}^\lambda * F_{\nu\lambda} = g_{\mu\nu}Y$, for the (anti-)self-duality configuration we have

$$T_{\mu\nu} = \frac{\sqrt{g}}{2} g_{\mu\nu} \left[L_F - X\frac{\partial L_F}{\partial X} - Y\frac{\partial L_F}{\partial Y} \right]_{X=\epsilon Y} . \tag{3.12}$$

Therefore, the vanishing of the energy-momentum tensor will hold if

$$\left[L_F - X\frac{\partial L_F}{\partial X} - Y\frac{\partial L_F}{\partial Y} \right]_{X=\epsilon Y} = 0 . \tag{3.13}$$

To avoid misunderstanding, we should remind the reader that not every regular solution of the equations of motion need (anti-)self-dual. We will give an explicit counter-example later.

3.2 Solution of the physical requirements

Now we summarize these three requirements on the form of the lagrangian

$$\text{(i)}\quad \text{Natural self-duality holds:}\ \left[1-\frac{\partial L_F}{\partial X}-\epsilon\frac{\partial L_F}{\partial Y}\right]_{X=\epsilon Y}=0\,. \tag{3.14}$$

$$\text{(ii)}\quad \text{Consistency with EOM:}\ \left[\frac{\partial^2 L_F}{\partial X^2}+2\epsilon\frac{\partial^2 L_F}{\partial X\partial Y}+\frac{\partial^2 L_F}{\partial Y^2}\right]_{X=\epsilon Y}=0\,. \tag{3.15}$$

$$\text{(iii)}\quad \text{Stress tensor vanishes:}\ \left[L_F-X\frac{\partial L_F}{\partial X}-Y\frac{\partial L_F}{\partial Y}\right]_{X=\epsilon Y}=0\,. \tag{3.16}$$

At first glance, these three conditions appear to be independent. However, in the following we show that the three conditions are equivalent with each other. Before considering the form of the solutions of the above conditions, we introduce two useful variables

$$u\equiv(X+\epsilon Y)/2\,,\qquad v\equiv(X-\epsilon Y)/2\,. \tag{3.17}$$

Using these variables, the above conditions are written as

$$\text{(i)}\quad \left.\frac{\partial L_F}{\partial u}\right|_{v=0}=1\,. \tag{3.18}$$

$$\text{(ii)}\quad \left.\frac{\partial^2 L_F}{\partial u^2}\right|_{v=0}=0\,. \tag{3.19}$$

$$\text{(iii)}\quad \left[L_F-u\frac{\partial L_F}{\partial u}\right]_{v=0}=0\,. \tag{3.20}$$

The lagrangian L_F may be written as a sum of homogeneous polynomials of u and v, so let us adopt the following expression for the lagrangian:

$$L_F=\sum_{m,n\geq 0}a_{mn}u^m v^n\,. \tag{3.21}$$

Now we assume that there is no cosmological term, $a_{00}=0$. Furthermore, in the weak field approximation, the lagrangian should be reduced to the Maxwell lagrangian, thus

$$L_F=X+\mathcal{O}\left(X^2,XY,Y^2\right)\,. \tag{3.22}$$

This implies that $a_{01}=a_{10}=1$. Under these assumptions, we can express the above three conditions in terms of the coefficient a_{mn}. It is easy to see that all of these conditions are equivalent with the following constraint:

$$a_{m0}=0\quad\text{for } m\geq 2\,. \tag{3.23}$$

Therefore we conclude that the above three conditions (i)–(iii) are equivalent with each other, and give the constraint (3.23) for the coefficients of the expansion of the lagrangian.

From a physical point of view, perhaps the simplest choice for the action is the one which gives duality-invariant equations of motion [7], [15]–[17]. Let us consider the compatibility of the SO(1, 1) duality[5] invariance requirement and the constraint above (3.23). The condition obtained in ref. [7] is

$$F_{\mu\nu} * F^{\mu\nu} = P_{\mu\nu} * P^{\mu\nu} \,. \tag{3.24}$$

When this condition is satisfied, the Euler-Lagrange equations for the gauge field are duality-invariant, and furthermore, the energy-momentum tensor is also duality-invariant. Thus the equation of motion for the gravity is also duality-invariant. In terms of our notation, the above condition is written as

$$\text{(iv)} \ \ \text{Duality invariance: } 2X\frac{\partial L_F}{\partial X}\frac{\partial L_F}{\partial Y} + Y\left\{\left(\frac{\partial L_F}{\partial X}\right)^2 + \left(\frac{\partial L_F}{\partial Y}\right)^2\right\} = Y \,. \tag{3.25}$$

Using u and v, this condition is simply expressed as

$$\text{(iv)} \ \ u\left(\frac{\partial L_F}{\partial u}\right)^2 - v\left(\frac{\partial L_F}{\partial v}\right)^2 = u - v \,. \tag{3.26}$$

Of course the Maxwell lagrangian $L_F = u + v$ satisfies this condition.

Putting $v = 0$ in this equation (3.26), then one observes that the condition (i) is deduced. Under some plausible assumption we have seen that three (i)–(iii) conditions are equivalent, thus consequently, we have shown that the duality invariant condition (iv) is sufficient for showing (i)–(iii). In sum, our conclusion is: *three conditions ((i) Validity of natural duality, (ii) Consistency of self-duality with EOM, (iii) vanishing of stress tensor) are equivalent with each other, and any duality invariant system possesses all of these three properties.*

As discussed in ref. [7], it is difficult to solve the condition (3.26) explicitly. For an example of duality invariant system, let us consider the Born-Infeld lagrangian

$$L_F = \frac{1}{\sqrt{g}}\sqrt{\det\left(g_{\mu\nu} + F_{\mu\nu}\right)} - 1 = \sqrt{1 + 2(u+v) + (u-v)^2} - 1 \,. \tag{3.27}$$

It is very easy to show that this lagrangian satisfies the above condition (3.26), and thus the results of the previous section are consistent with the argument presented in this section. See ref. [14] for some other solutions.

4. Higher-rank generalization

In the previous sections, we have seen some interesting properties of the self-dual configurations and their relation to the duality-invariant system. Now in this section, we treat the case in which there are several U(1) gauge fields. As for non-abelian gauge groups, comments will be given in the discussion.

[5]The duality group is now SO(1, 1), not SO(2). This is due to the fact that we are in euclidean regime. See section 5.1.

4.1 Three conditions and their compatibility

Among many possibilities for the lagrangian of n gauge fields $A_\mu^{(i)}$ where $i = 1, \ldots, n$, we adopt the following form of the lagrangian

$$L_F = L_F(X_i, Y_i) \tag{4.1}$$

which is a simple generalization[6] of (3.6). Here the Lorentz-invariant quantities are defined as

$$X_i \equiv F_{\mu\nu}^{(i)} F^{(i)\mu\nu}/4\,, \qquad Y_i \equiv F_{\mu\nu}^{(i)} * F^{(i)\mu\nu}/4\,. \tag{4.2}$$

The dual field strength is also generalized as

$$P^{(i)\mu\nu} \equiv 2\frac{\partial L_F}{\partial F^{(i)\mu\nu}} = \left(F^{(i)\mu\nu}\frac{\partial L_F}{\partial X_i} + *F^{(i)\mu\nu}\frac{\partial L_F}{\partial Y_i} \right). \tag{4.3}$$

Using this lagrangian, it is easy to derive the three conditions in the previous section. The first requirement, which is the consistency with the natural self-duality condition with the usual self-duality condition, is expressed by

$$\left[1 - \frac{\partial L_F}{\partial X_i} - \epsilon_i \frac{\partial L_F}{\partial Y_i} \right]_{X_j = \epsilon_j Y_j} = 0\,. \tag{4.4}$$

In this expression, we have used a constant ϵ_i which equals to ± 1, depending on whether the gauge field $A_\mu^{(i)}$ is self-dual or anti-self-dual. This condition must be understood to hold for arbitrary i. The second requirement of the consistency with equations of motion reads

$$\left[\frac{\partial^2 L_F}{\partial X_i \partial X_j} + \epsilon_j \frac{\partial^2 L_F}{\partial X_i \partial Y_j} + \epsilon_i \frac{\partial^2 L_F}{\partial Y_i \partial X_j} + \epsilon_i \epsilon_j \frac{\partial^2 L_F}{\partial Y_i \partial Y_j} \right]_{X_k = \epsilon_k Y_k \ \text{for any}\ k} = 0\,, \tag{4.5}$$

for arbitrary i and j. The third condition of the vanishing of the energy-momentum tensor is

$$\left[L_F - \sum_i \left(X_i \frac{\partial L_F}{\partial X_i} + Y_i \frac{\partial L_F}{\partial Y_i} \right) \right]_{X_j = \epsilon_j Y_j} = 0\,. \tag{4.6}$$

Let us make a change of variables in a manner similar to the previous section as

$$u_i \equiv (X_i + \epsilon_i Y_i)/2\,, \qquad v_i \equiv (X_i - \epsilon_i Y_i)/2\,. \tag{4.7}$$

Then the three conditions become

$$\text{(i)} \quad \left. \frac{\partial L_F}{\partial u_i} \right|_{v_j = 0} = 1\,, \tag{4.8}$$

$$\text{(ii)} \quad \left. \frac{\partial^2 L_F}{\partial u_i \partial u_j} \right|_{v_k = 0} = 0\,. \tag{4.9}$$

$$\text{(iii)} \quad \left[L_F - \sum_i u_i \frac{\partial L_F}{\partial u_i} \right]_{v_j = 0} = 0\,. \tag{4.10}$$

[6] We have assumed that there is no cross term such as $F_{\mu\nu}^{(1)} * F^{(2)\mu\nu}$.

How these three conditions are compatible? Noting that especially the third condition (4.6) is only a single condition whereas the first and the second conditions are more than that, it is obvious that these three conditions are not equivalent with each other. This is in contrast with the case of a single gauge field. However, as we shall see, condition (i) is equivalent to condition (ii). Let us assume that the lagrangian is a polynomial of u_i and v_i:

$$L_F = \sum_{k_1 k_2 \cdots l_n} a_{k_1 k_2 \cdots l_n} \Pi_{i=1}^n u_i^{k_i} \Pi_{i=1}^n v_i^{l_i} \,. \tag{4.11}$$

Then the condition (i) implies

$$L_F = \sum_i (u_i + v_i) + O(uv) \,. \tag{4.12}$$

Here of course we assumed that the cosmological constant is zero. This behavior is consistent with the requirement that the lagrangian must reduce to the Maxwell system in the weak field limit. It is easy to see that eq. (4.12) is equivalent with the condition (ii) if we adopt the form of the weak field limit. The condition (iii) is, however, a weaker one compared to (i) and (ii). A simple calculation show that the form (4.12) satisfies the condition (iii). Thus, we conclude that, under the appropriate weak-field assumption,

$$\text{(i)} \Longleftrightarrow \text{(ii)} \Longrightarrow \text{(iii)} \,. \tag{4.13}$$

4.2 Duality-invariant systems

The simple choice for a lagrangian is the one which gives duality-invariant equations of motion as seen in the previous section. Now we have several gauge fields, therefore there are various versions of duality rotations. First, let us generalize the argument given in ref. [7] and refs. [15, 16].

The most general duality invariance is defined as [17]

$$\delta \begin{pmatrix} *P \\ F \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} *P \\ F \end{pmatrix} , \tag{4.14}$$

where A, B, C, D are $n \times n$ matrices. Only when we include scalar fields which transform appropriately under the duality, the full duality group $\mathrm{Sp}(2n, \mathbb{R})$ is obtained. However we are studying a system with only the gauge fields, the full available duality group is $\mathrm{U}(n)$ which is the maximal compact subgroup of $\mathrm{Sp}(2n, \mathbb{R})$ (see section 5.3 for the argument with adding scalar fields and enhancement of the symmetry). This is obtained by imposing

$$A = D = -A^T \,, \qquad B = -C = B^T \,. \tag{4.15}$$

(In this subsection we are temporarily working in the lorentzian signature adopted in ref. [7].)

Let us consider the general constraint on the constitutive relation (4.3) under the requirement of this duality rotation invariance, following ref. [7]. Performing the duality rotation in the both sides of eq. (4.3), one has

$$2A^{ik}\frac{\partial L}{\partial F^{(k)\mu\nu}} + B^{ik} * F^{(k)\mu\nu} = 2\left[-2B^{jm}\left(*\frac{\partial L}{\partial F^{(m)}_{\rho\sigma}}\right) + A^{jm}F^{(m)\rho\sigma}\right] \times \\ \times \frac{\partial}{\partial F^{(j)\rho\sigma}}\left(\frac{\partial L}{\partial F^{(i)\mu\nu}}\right). \tag{4.16}$$

If we demand that this equation is satisfied for any anti-symmetric matrix A and any symmetric matrix B, then the full compact duality group $\mathrm{U}(n)$ is obtained. Since A and B are independent of each other, we have

$$A^{ik}\frac{\partial L}{\partial F^{(k)\mu\nu}} = A^{jm}F^{(m)\rho\sigma}\frac{\partial}{\partial F^{(j)\rho\sigma}}\frac{\partial L}{\partial F^{(i)\mu\nu}}\,, \tag{4.17}$$

and

$$B^{ik} * F^{(k)\mu\nu} = -4B^{jm}\left(*\frac{\partial L}{\partial F^{(m)}_{\rho\sigma}}\right)\frac{\partial}{\partial F^{(j)\rho\sigma}}\left(\frac{\partial L}{\partial F^{(i)\mu\nu}}\right). \tag{4.18}$$

For the case of $n = 1$ as in ref. [7], there is no antisymmetric 1×1 matrix, so $A = 0$ holds. Then the first condition does not appear. For general n, it turns out that the first condition (4.17) is equivalent with the invariance of the lagrangian under the rotation defined by A. Using a relation

$$F^{(m)\rho\sigma}\frac{\partial}{\partial F^{(j)\rho\sigma}}\frac{\partial L}{\partial F^{(i)\mu\nu}} = \frac{\partial}{\partial F^{(i)\mu\nu}}\left(F^{(m)\rho\sigma}\frac{\partial L}{\partial F^{(j)\rho\sigma}}\right) - \delta^{im}\frac{\partial L}{\partial F^{(j)\mu\nu}}\,, \tag{4.19}$$

we see that the condition (4.17) reduces to[7]

$$A^{jm}F^{(m)\rho\sigma}\frac{\partial L}{\partial F^{(j)\rho\sigma}} = 0\,. \tag{4.20}$$

This is merely a condition of the invariance of the lagrangian. So if one wants to know whether a given lagrangian has symmetry generated by the matrix A, then one has only to check the invariance of the lagrangian under the transformation $F^{(i)} \to A^{ij}F^{(j)}$. The largest symmetry associated with A is $\mathrm{SO}(n)$.

The second condition for B (4.18) can be integrated in a similar manner to ref. [7], and the result is

$$F^{(i)}_{\mu\nu}B^{ik} * F^{(k)\mu\nu} = -B^{jm}\left(2\frac{\partial L}{\partial F^{(m)}_{\rho\sigma}}\right)\left(2 * \frac{\partial L}{\partial F^{(i)\rho\sigma}}\right). \tag{4.21}$$

Here we have neglected the integration constant which violates the invariance of the energy-momentum tensor. This condition is equivalent with

$$B^{ik}\left(F^{(i)}_{\mu\nu} * F^{(k)\mu\nu} + P^{(i)}_{\mu\nu} * P^{(k)\mu\nu}\right) = 0\,. \tag{4.22}$$

[7] We ignore the integration constant because it will not appear in general.

This is actually the generalization of the condition for $n = 1$ in refs. [7, 15, 16]. The duality group of a system can be read from this equation by checking which form of B satisfies the equation (4.22) when a certain lagrangian is given.

For example, if we restrict the form of B as

$$B = 1_n \,, \tag{4.23}$$

where 1_n is a unit $n \times n$ matrix, then this indicates that the system satisfying the conditions (4.22) and (4.23) possesses only the symmetry involved with simultaneous duality rotation for all the gauge fields. The corresponding condition in the euclidean signature is

$$\sum_i \left(F^{(i)}_{\mu\nu} * F^{(i)\mu\nu} - P^{(i)}_{\mu\nu} * P^{(i)\mu\nu} \right) = 0 \,. \tag{4.24}$$

We intend to check how the the simultaneous duality invariance (4.24), which is expressed as

$$\sum_i \left[u_i \left(\frac{\partial L_F}{\partial u_i} \right)^2 - v_i \left(\frac{\partial L_F}{\partial v_i} \right)^2 - (u_i - v_i) \right] = 0 \,, \tag{4.25}$$

is consistently reproducing the three conditions (4.8)–(4.10). However, it is obvious that this duality invariance condition is not sufficient to reproduce them. This is because the number of the conditions is not sufficient.

On the other hand, we can require a rather high symmetry such as

$$\{B\} = \Big\{ \mathrm{diag}(0, \ldots, 0, 1, 0, \ldots, 0) \mid i = 1, \ldots, n \Big\} \,, \tag{4.26}$$

where the diagonal matrix denotes the one in which only the i-th diagonal entry is non vanishing. Then of course we can derive a much restrictive condition

$$F^{(i)}_{\mu\nu} * F^{(i)\mu\nu} - P^{(i)}_{\mu\nu} * P^{(i)\mu\nu} = 0 \tag{4.27}$$

for any i. In this case the duality group from the B part is $(\mathrm{SO}(2))^n$. From this condition it is possible to derive all the three conditions (i)–(iii).

From the above argument, we have seen that the full antisymmetric B is not necessary for deriving the three conditions. Only the form (4.26) is necessary.

4.3 Some examples

In the following, we present explicit examples of the generalization of the Born-Infeld action to the higher rank case. We see, however, that an intriguing example (denoted as example 3 below) satisfies all the three conditions in spite of the fact that the lagrangian does not satisfy the strong condition (4.27).

4.3.1 Example 1: Effective action of string theory

The abelian Born-Infeld lagrangian admits D-brane interpretation in string theory. This theory describes low energy effective theory on a single D-brane. “Low energy” means the slowly-varying field approximation, where the derivative of the field strength is neglected. A straightforward generalization of this D-brane action to the higher rank case may be obtained in the situation where many parallel D-branes are considered. Naively in this case the low energy description is associated with non-abelian gauge groups. One of the proposed non-abelian non-linear lagrangian describing this system is [18]

$$L_F = \mathrm{STr}\sqrt{\det\left(g_{\mu\nu} + F_{\mu\nu}\right)} - \sqrt{g}\,, \tag{4.28}$$

where the gauge fields are non-abelian matrices, and the the symmetric trace operation is normalized as

$$\mathrm{STr}\, T^a T^b = \delta^{ab}\,. \tag{4.29}$$

If we simply restrict the configuration of the gauge fields to only the diagonal entries, then the resultant lagrangian is the summation of the abelian Born-Infeld lagrangian. This situation can be achieved in string theory when the distance between each brane becomes large compared to the string scale.

$$L_F = \sum_{i=1}^{n}\left(\sqrt{\det\left(g_{\mu\nu} + F^{(i)}_{\mu\nu}\right)} - \sqrt{g}\right). \tag{4.30}$$

Obviously, this lagrangian satisfies the duality invariance condition with (4.26), since the lagrangian is merely a summation of the abelian Born-Infeld lagrangian. Thus this lagrangian has an $(\mathrm{SO}(2))^n$ duality group. In other words each U(1) field may be subjected to an independent SO(2) duality rotation defined on each D-brane. Of course the lagrangian satisfies the three criteria (4.8)–(4.10). As for the A part, there remains S_n symmetry permuting the n U(1)s.

To be summarized, this example corresponds to a duality matrix restricted to

$$A = 0\,, \qquad \{B\} = \left\{\mathrm{diag}(0,\ldots,0,1,0,\ldots,0) \mid i = 1,\ldots,n\right\}. \tag{4.31}$$

The duality group is $(\mathrm{SO}(2))^n$ but there is no global continuous symmetry which rotates the label of the gauge fields.

There is a topological bound for this system. The bound is realized when the gauge fields are (anti-)self-dual, but one can choose self-dual or anti-self-dual independently for each index (i). However, in order to preserve the target space supersymmetry in the D-brane sense, the configuration must be all self-dual or all anti-self-dual. The other configurations may break supersymmetry.

4.3.2 Example 2: A frustrated topological bound

In ref. [19], the following lagrangian was suggested in the flat space:

$$L_F = \sqrt{n\left(n + 2\sum_i X_i + \sum_i Y_i^2\right)} - n\,. \tag{4.32}$$

This corresponds to taking the symmetrized trace inside the square root, and thus seems to be unrelated to string theory, as pointed out in ref. [3]. There is a topological lower bound for the action but it is easy to see that it can be attained only when all the U(1) gauge fields are equal to each other. (The system is "frustrated".) This is reflected in the fact that this lagrangian satisfies none of the three criteria (4.8)–(4.10). Specifically,

$$L_F \geq \sqrt{n\sum_i (1 \pm Y_i)^2} - n\,. \tag{4.33}$$

The bound is saturated if $F^{(i)}_{\mu\nu} = \pm * F^{(i)}_{\mu\nu}$ for any i, that is, when the individual U(1) fields are (anti-)self-dual. The right hand side of eq. (4.33) is certainly bigger than the topological quantity

$$L_F \geq \pm\sum_i Y_i\, a. \tag{4.34}$$

To derive this fact one uses the following inequality:

$$n\sum_i a_i^2 = \sum_{i<j}(a_i - a_j)^2 + \left(\sum_i a_i\right)^2 \geq \left(\sum_i a_i\right)^2. \tag{4.35}$$

Thus although the topological inequality (4.34) holds if the n quantities $\mathbf{B}^{(i)} \cdot \mathbf{E}^{(i)}$ on the right-hand-side are unequal, nevertheless, the topological bound (4.34) can be saturated if and only if

$$F^{(1)} = \pm * F^{(1)} = F^{(2)} = \pm * F^{(2)} = F^{(3)} = \pm * F^{(3)} = \cdots. \tag{4.36}$$

In this sense, the action is bounded below by the action of the strictly constrained configurations. We cannot attain the bound in a component-wise fashion.

4.3.3 Example 3: A new model with large symmetry

In this subsection, we introduce a remarkable model with the SO(2) duality group and SO(n) global symmetry. One of the lagrangians which have large duality symmetry was constructed by Zumino et al. [20, 21]. Our lagrangian is different from theirs. We do not know the connection with string theory.

JHEP09(2000)013

Let us consider a duality invariant euclidean action of the following form:

$$
\begin{aligned}
L_F &= \sqrt{1 + 2\sum_i X_i + \left(\sum_i Y_i\right)^2} - 1 \\
&= \sqrt{1 + 2\sum_i (u_i + v_i) + \left(\sum_i u_i - v_i\right)^2} - 1\,.
\end{aligned} \tag{4.37}
$$

This lagrangian satisfies the above simultaneous duality invariance condition (4.24). This is a simple generalization of the abelian Born-Infeld lagrangian.[8] The system described by this lagrangian possesses a global $\mathrm{SO}(n)\times\mathrm{SO}(1,1)_{\mathrm{elemag}}$ symmetry. The $\mathrm{SO}(n)$ rotates the label of the gauge field. The $\mathrm{SO}(1,1)$ is the simultaneous electromagnetic duality rotation on all the gauge fields. In terms of the matrices A and B in the previous subsection, our action seems to correspond to

$$
\text{arbitrary anti-symmetric } A\,, \qquad B = 1_n \tag{4.38}
$$

which indicates that we have the duality group $\mathrm{SO}(n)\times\mathrm{SO}(2)$.

Since the action (4.37) does not satisfy a restrictive duality invariance condition (4.27), it is expected that the lagrangian does not possess the three properties (i)–(iii). However, surprisingly, it is a straightforward calculation to show that this lagrangian satisfies all the three conditions (i)–(iii). So this is one non-trivial example which attain many desirable properties even in the curved background.

5. Aspects of general non-linear electrodynamics

In this section we illustrate various aspects of non-linear electrodynamics by taking concrete examples including the example 3 of section 4.3.3. We shall work in a flat manifold for simplicity in sections 5.1, 5.2 and 5.3.

5.1 Hamiltonian for the $\mathrm{SO}(n)\times\mathrm{SO}(2)$ symmetric system

Let us explore the properties of the new model considered above in example 3. To see the symmetry of the system, the hamiltonian formalism is useful. The lorentzian lagrangian is

$$
L_F = 1 - \sqrt{1 + \sum_i \left(-|\mathbf{E}^{(i)}|^2 + |\mathbf{B}^{(i)}|^2\right) - \left(\sum_i \mathbf{E}^{(i)}\cdot\mathbf{B}^{(i)}\right)^2}\,. \tag{5.1}
$$

[8] This lagrangian is not the well-known non-abelian Born-Infeld lagrangian with only the diagonal entries, as seen in comparison with the lagrangian of section 4.3.1.

Performing a Legendre transformation as in ref. [22], we can construct the hamiltonian as a function of $\mathbf{B}^{(i)}$ and $\mathbf{D}^{(i)}$. We find that

$$H=\sqrt{\left(1+\sum_i|\mathbf{D}^{(i)}|^2\right)\left(1+\sum_i|\mathbf{B}^{(i)}|^2\right)-\left(\sum_i\mathbf{D}^{(i)}\cdot\mathbf{B}^{(i)}\right)^2}-1\,. \qquad (5.2)$$

This expression is manifestly invariant under $\mathrm{SO}(2)\times\mathrm{SO}(n)$ where the SO(2) rotates $\mathbf{B}^{(i)}$ into $\mathbf{D}^{(i)}$.

In the case $n=2$, the lagrangian may be written in the form

$$L_F=\sqrt{1+2\alpha-\beta^2}-1\,, \qquad (5.3)$$

where $\alpha\equiv F_{\mu\nu}\bar{F}^{\mu\nu}/4$ and $\beta\equiv F_{\mu\nu}*\bar{F}^{\mu\nu}/4$. We have introduced two real gauge fields and combine them to a single complex gauge field. This lagrangian is the one given in refs. [20, 21].

Using the method of ref. [23], one may extend the compact SO(2) duality group to the non-compact group $\mathrm{Sp}(2,\mathbb{R})(=\mathrm{SL}(2,\mathbb{R}))$ by adding an axion and dilaton field which will be done in section 5.3.

When dealing with time-independent fields, it is convenient to introduce potentials. We must first perform a Legendre transformation to get an action in terms of $\mathbf{E}$ and $\mathbf{H}$, and then substitute $\mathbf{E}=\nabla\phi$, $\mathbf{H}=\nabla\chi$. We then get the $\mathrm{SO}(n)\times\mathrm{SO}(2)$ invariant energy functional

$$\hat{L}_F=\sqrt{\left(1-\sum_i|\nabla\phi^{(i)}|^2\right)\left(1-\sum_i|\nabla\chi^{(i)}|^2\right)-\left(\sum_i\nabla\phi^{(i)}\cdot\nabla\chi^{(i)}\right)^2}-1\,. \qquad (5.4)$$

The dimensional reduction is a useful method to realize the duality symmetry of the system. Surprisingly, this dimensional reduction can be performed even in the curved space, which will be studied in section 5.4.

5.2 Legendre self-duality

The SO(2)-invariance of the equations of motion does not mean that the lagrangian is invariant but it does mean that the quantity

$$L+\frac{1}{4}\sum P^{(i)\mu\nu}F^{(i)}_{\mu\nu} \qquad (5.5)$$

is invariant, with

$$P^{(i)\mu\nu}=-2\frac{\partial L}{\partial F^{(i)}_{\mu\nu}} \qquad (5.6)$$

(we are using conventions appropriate for Minkowski signature). As pointed out by Gaillard and Zumino [15] this gives rise to a number of identities including

$$L\left(*P^{(i)\mu\nu}\right)=L\left(F^{(i)}_{\mu\nu}\right)+\frac{1}{2}\sum_i F^{(i)}_{\mu\nu}\frac{\partial L}{\partial F^{(i)}_{\mu\nu}}\,. \qquad (5.7)$$

They also pointed out that identity (5.7) may be interpreted in terms of a version of Legendre duality interchanging the Bianchi identity $d*F^{(i)}=0$ and equations of motion $d*P^{(i)}=0$. Locally the former is equivalent to $F=dA^{(i)}$, while locally the latter is equivalent to $*P^{(i)}=dB^{(i)}$. Assuming that the Bianchi identity is given, the field equation may obtained by varying the lagrangian considered as a function of dA, $L=L(dA)$. Dually one could assume the field equation and obtain the Bianchi identity from a dual lagrangian $L_{\mathrm{D}}(dB)$.

The two variational principles and the two lagrangians are related by a Legendre transformation, as can be seen by passing to first order formalism. One considers the lagrangian

$$L(F,B)=L\left(F^{(i)}\right)-\frac{1}{2}\sum_i *F^{(i)\mu\nu}\left(\partial_\mu B^{(i)}_\nu-\partial_\nu B^{(i)}_\mu\right). \tag{5.8}$$

Variation with respect to $B^{(i)}_\mu$ gives the Bianchi identity $d*F=0$. Variation with respect $F^{(i)}_{\mu\nu}$ gives the field equation $*P^{(i)}=dB^{(i)}$. Eliminating $F^{(i)}$ in favor of $P^{(i)}$ gives $L_{\mathrm{D}}(dB^{(i)})$. Thus

$$L_{\mathrm{D}}\left(*P^{(i)}\right)=L\left(F^{(i)}\right)+\frac{1}{2}\sum_i P^{(i)\mu\nu}F^{(i)}_{\mu\nu}\,, \tag{5.9}$$

in other words

$$L_{\mathrm{D}}\left(*P^{(i)}\right)=-\hat{L}\left(P^{(i)\mu\nu}\right), \tag{5.10}$$

where $\hat{L}(P^{(i)})$ is the Legendre transform of $L(F^{(i)})$

$$\hat{L}\left(P^{(i)}\right)=-\frac{1}{2}\sum P^{(i)\mu\nu}F^{(i)}_{\mu\nu}-L\left(F^{(i)}\right) \tag{5.11}$$

such that

$$F^{(i)}_{\mu\nu}=-2\frac{\partial\hat{L}}{\partial P^{(i)\mu\nu}}\,. \tag{5.12}$$

Of course one could have started with the first order system

$$L_{\mathrm{D}}\left(*P^{(i)},A\right)=L_{\mathrm{D}}\left(*P^{(i)}\right)-\frac{1}{2}\sum *P^{(i)\mu\nu}\left(\partial_\mu A^{(i)}{}_\nu-\partial_\nu A^{(i)}{}_\mu\right). \tag{5.13}$$

Variation with respect to $A^{(i)}_\mu$ yields the field equation $d*P^{(i)}=0$, and variation with respect to $*P^{(i)}$, using (5.12) gives the Bianchi identity $d*F^{(i)}=0$.

For a general non-linear electrodynamic lagrangian the two lagrangians $L_{\mathrm{D}}(*P^{(i)})$ and $L(F^{(i)})$ are unrelated, but for a theory invariant under SO(2) duality rotations (5.7) implies that

$$L_{\mathrm{D}}(*P^{(i)})=L(*P^{(i)})\,. \tag{5.14}$$

In our case of example 3, we have $*P^{(i)}=(\mathbf{H}^{(i)},-\mathbf{D}^{(i)})$, and the system has SO(2) invariance as seen before, then the Legendre-transformed lagrangian is in the same form as

$$\hat{L} = \sqrt{1 - \sum(|\mathbf{H}^{(i)}|^2 - |\mathbf{D}^{(i)}|^2) - \left(\sum \mathbf{H}^{(i)} \cdot \mathbf{D}^{(i)}\right)^2} - 1\,. \tag{5.15}$$

Technically this property of the form of the lagrangian is understood as follows. First we have lagrangian $L(\mathbf{E}, \mathbf{B})$. From this lagrangian we obtain hamiltonian $H(\mathbf{D}, \mathbf{B})$ by Legendre transformation. Further dualizing $\mathbf{B}$ into $\mathbf{H}$, then one has perfectly dual description $\hat{L}(\mathbf{D}, \mathbf{H})$. Now, if the system has SO(2) duality invariance, then the hamiltonian is invariant under the exchange $\mathbf{D} \leftrightarrow \mathbf{B}$. So the second Legendre transformation is precisely the same calculation of the first Legendre transformation, and one gets the same form as the starting point L.

To complete this section we wish to point out that Legendre self-duality does not imply the existence of a continuous SO(2) duality invariance. As a counter-example one may take the theory originally considered by Born, which has $L = 1 - \sqrt{1 + 2X}$. A simple calculation reveals that this theory is Legendre self-dual but also that it does not satisfy the necessary condition for SO(2)-invariance.

5.3 Adding dilaton and axion

To couple to an axion a and a dilaton Φ we follow a general procedure which we shall illustrate by means of a particular example (model 3 of section 4.3.3). We choose to work in the hamiltonian formalism and shall exhibit a manifestly $\mathrm{Sp}(2, \mathbb{R})$-invariant hamiltonian. The covariant formalism is equally simple and follows closely the discussion for a single U(1) given in ref. [23]. We begin with inserting appropriate factors of e^{Φ} into our original action, adding a "θ term" coupling to the axion and further adding an $\mathrm{Sp}(2, \mathbb{R})$-invariant kinetic action for the axion and dilaton

$$L = L_{\mathrm{NLE}}\left(e^{\Phi}\mathbf{E}^{(i)}, e^{\Phi}\mathbf{B}^{(i)}\right) - a \sum \left(\mathbf{E}^{(i)} \cdot \mathbf{B}^{(i)}\right) + L_{\mathrm{AD}}(\Phi, a)\,. \tag{5.16}$$

The action L_{NLE} is now invariant under dilations of $\mathbf{E}^{(i)}$ and $\mathbf{B}^{(i)}$ provided one compensates with an appropriate shift of the dilaton. We are naturally led to introduce calligraphic variables which are invariant under dilations: $\mathcal{E}^{(i)} \equiv e^{\Phi}\mathbf{E}^{(i)}$ and $\mathcal{B}^{(i)} \equiv e^{\Phi}\mathbf{B}^{(i)}$.

We now calculate the electric inductions $\mathbf{D}^{(i)}$ and find that they are changed both by a shift and a scaling

$$\mathbf{D}^{(i)} = e^{\Phi}\mathbf{D}^{(i)}\left(\mathcal{E}^{(i)}, \mathcal{B}^{(i)}\right) - a\mathbf{B}^{(i)}\,, \tag{5.17}$$

where $\mathbf{D}^{(i)}(\mathcal{E}^{(i)}, \mathcal{B}^{(i)})$ is the constitutive relation in the absence of the axion and dilaton but regarded as a function of the calligraphic variables. Clearly $\mathbf{D}^{(i)}(\mathcal{E}^{(i)}, \mathcal{B}^{(i)})$ is invariant under a shift of the dilaton, since it is only a function of the calligraphic variables which by construction are invariant and therefore $e^{\phi}\mathbf{D}^{(i)}(\mathcal{E}^{(i)}, \mathcal{B}^{(i)})$ will take on the same factor as e^{Φ}. Since $\mathbf{B}^{(i)}$ takes on the same factor as $e^{-\Phi}$ we need the axion a to take on the same factor as $e^{2\Phi}$ to make all terms on the the right-hand side of (5.17) scale as e^{Φ}. This fixes up to an over all multiple the axion-dilaton lagrangian $L_{\rm AD}$. Now $\mathbf{D}^{(i)}$ will scale[9] as e^{Φ}. We therefore define calligraphic inductions by

$$\mathcal{D}^{(i)} \equiv e^{-\Phi}\left(\mathbf{D}^{(i)} + a\mathbf{B}^{(i)}\right). \tag{5.18}$$

By construction $\mathcal{D}^{(i)}$ will be invariant under shifts of the dilaton. Moreover under the shift of the axion $a \to a + \alpha$, $\mathcal{D}^{(i)}$ will be also be invariant as long as $\mathbf{D}^{(i)} \to \mathbf{D}^{(i)} - \alpha\mathbf{B}^{(i)}$.

Explicitly, in our example 3,

$$\mathbf{D}^{(i)} + a\mathbf{B}^{(i)} = \frac{e^{2\Phi}\mathbf{E}^{(i)} + e^{4\Phi}\sum_j(\mathbf{E}^{(j)}\cdot\mathbf{B}^{(j)})\mathbf{B}^{(i)}}{\sqrt{1 - e^{2\Phi}\sum_k\left(|\mathbf{E}^{(k)}|^2 - |\mathbf{B}^{(k)}|^2\right) - e^{4\Phi}\left(\sum_k \mathbf{E}^{(k)}\cdot\mathbf{B}^{(k)}\right)^2}}. \tag{5.19}$$

Thus

$$\mathcal{D}^{(i)} = \frac{\mathcal{E}^{(i)} + \sum_j\left(\mathcal{E}^{(j)}\cdot\mathcal{B}^{(j)}\right)\mathcal{B}^{(i)}}{\sqrt{1 - \sum_k\left(|\mathcal{E}^{(k)}|^2 - |\mathcal{B}^{(k)}|^2\right) - \left(\sum_k \mathcal{E}^{(k)}\cdot\mathcal{B}^{(k)}\right)^2}} \tag{5.20}$$

which is identical, in terms of the calligraphic variables, to the constitutive relation in the absence of the axion and dilaton.

Now one finds that

$$H = H_{\rm NLE} + H_{\rm AD}\,, \tag{5.21}$$

where $H_{\rm AD}$ is the hamiltonian of the axion and dilaton fields constructed from $L_{\rm AD}$ and which is manifestly $\mathrm{Sp}(2,\mathbb{R})$-invariant. And in our example

$$H_{\rm NLE} + 1 = \frac{1 + \sum|\mathcal{B}^{(i)}|^2}{\sqrt{1 - \sum_j\left(|\mathcal{E}^{(j)}|^2 - |\mathcal{B}^{(j)}|^2\right) - \left(\sum_j \mathcal{E}^{(j)}\cdot\mathcal{B}^{(j)}\right)^2}}. \tag{5.22}$$

Thus in terms of the calligraphic variables, $H_{\rm NLE}$ has the same functional form as it does in terms of the usual variables in the absence of the axion and dilaton. We may use the constitutive relation to express $H_{\rm NLE}$ as a function of and $\mathcal{B}^{(i)}$ and $\mathcal{D}^{(i)}$. It will be identical to the expression in the absence of the axion and dilaton. In our case

$$H_{\rm NLE} + 1 = \sqrt{\left(1 + \sum_i|\mathcal{B}^{(i)}|^2\right)\left(1 + \sum_j|\mathcal{D}^{(j)}|^2\right) - \left(\sum_k \mathcal{B}^{(k)}\cdot\mathcal{D}^{(k)}\right)^2}. \tag{5.23}$$

It is clear that the resulting hamiltonian is invariant under shifts of the dilaton and axion, and this is a general result. What about the $\mathrm{SO}(2) \subset \mathrm{Sp}(2,\mathbb{R})$ duality rotations? In principle it is now possible to express $H_{\rm NLE}$ in terms of the variables $\mathbf{B}^{(i)}$ and $\mathbf{D}^{(i)}$ but this is complicated and unnecessary. Moreover we do not at this

[9] Note that we distinguish between $\mathbf{D}^{(i)}$ and $\mathbf{D}^{(i)}(\mathcal{E}^{(i)}, \mathcal{B}^{(i)})$.

stage know the appropriate transformation rule in terms of these variables. Of course we do know how SO(2) acts of the axion and dilaton so that L_{AD} and hence H_{AD} is invariant. Moreover if the original theory had $H_{\mathrm{NLE}}(\mathbf{B}^{(i)}, \mathbf{D}^{(i)})$ invariant under SO(2) rotations of $\mathbf{B}^{(i)}$ into $\mathbf{D}^{(i)}$ then $H_{\mathrm{NLE}}(\mathcal{B}^{(i)}, \mathcal{D}^{(i)})$ will also be invariant under SO(2) rotations of $\mathcal{B}^{(i)}$ into $\mathcal{D}^{(i)}$. This is certainly the case of our model 3. It is clear that the action of SO(2) is the standard one on the calligraphic variables and from this we may work out the transformation rule on the original variables.

One way of viewing this construction is to note that the variables $\mathbf{B}^{(i)}$ into $\mathbf{D}^{(i)}$ are canonically conjugate and the passage to the variables $\mathcal{B}^{(i)}$ into $\mathcal{D}^{(i)}$ is a canonical transformation.

5.4 Dimensional reduction

It is known that in flat space dimensional reduction of the Born-Infeld action leads to an effective action in which duality invariance is manifest. In this subsection we show that the same is true in curved spacetime. We assume a stationary lorentzian metric which locally at least is defined on $M \equiv \mathbb{R} \times \Sigma$ and takes the form

$$ds^2 = -e^{2U}\left(dt + \omega_i dx^i\right)^2 + e^{-2U}\gamma_{ij}dx^i dx^j \tag{5.24}$$

and we consider a stationary U(1) field

$$A = \phi\left(dt + \omega_i dx^i\right) + \hat{A}_i dx^i \,. \tag{5.25}$$

From now on we consider all quantities as living on Σ equipped with the metric γ_{ij}. Thus all inner products, covariant derivatives etc. are taken with respect to the metric γ_{ij}. We have

$$F = d\phi \wedge (dt + \omega) + d\hat{A} + \phi d\omega \,, \tag{5.26}$$

where $\omega \equiv \omega_i dx^i$. The electric field is $\mathbf{E} = -\nabla\phi$ and we define the magnetic field by

$$\mathbf{B} = *_\gamma\left(d\hat{A} + \phi d\omega\right). \tag{5.27}$$

Thus

$$\nabla \cdot \mathbf{B} = \nabla\phi \cdot \mathbf{\Omega} \,, \tag{5.28}$$

where $\mathbf{\Omega} = *_\gamma d\omega$. The two invariants are given as

$$F_{\mu\nu}F^{\mu\nu} = 2\left(-\nabla\phi \cdot \nabla\phi + e^{4U}\mathbf{B}\cdot\mathbf{B}\right) \tag{5.29}$$

and

$$F_{\mu\nu} * F^{\mu\nu} = 4e^{2U}\nabla\phi \cdot \mathbf{B} \,. \tag{5.30}$$

Using the fact that $\sqrt{-g} = e^{-2U}\sqrt{\gamma}$ we find that

$$\sqrt{-g}F_{\mu\nu} * F^{\mu\nu} = 4\sqrt{\gamma}\nabla\phi \cdot \mathbf{B} = \nabla\left(4\sqrt{\gamma}\left(\phi\nabla \times A + \frac{1}{2}\phi^2\mathbf{\Omega}\right)\right), \tag{5.31}$$

which is a total derivative as expected.

The Born-Infeld lagrangian is now

$$\sqrt{\gamma}e^{-2U}\left(1-\sqrt{1-\nabla\phi\cdot\nabla\phi+e^{4U}\mathbf{B}\cdot\mathbf{B}-\left(e^{2U}\nabla\phi\cdot\mathbf{B}\right)^2}\right). \tag{5.32}$$

We must introduce a Lagrange multiplier χ to enforce the constraint (5.28). Thus we add

$$\chi\left(\nabla\cdot\mathbf{B}-\nabla\phi\cdot\mathbf{\Omega}\right)\sqrt{\gamma}\,. \tag{5.33}$$

Integration by parts and variation with respect to $\mathbf{B}$ yields

$$-e^{-2U}\nabla\chi=\frac{\mathbf{B}-\nabla\phi(\mathbf{B}\cdot\nabla\phi)}{\sqrt{1-\nabla\phi\cdot\nabla\phi+e^{4U}\mathbf{B}\cdot\mathbf{B}-\left(e^{2U}\nabla\phi\cdot\mathbf{B}\right)^2}}\,. \tag{5.34}$$

One now substitutes back into the lagrangian to get

$$\sqrt{\gamma}e^{-2U}\left(1-\sqrt{(1-(\nabla\phi)^2)\,(1-(\nabla\chi)^2)-(\nabla\phi\cdot\nabla\chi)^2}\right)-\chi\nabla\phi\cdot\mathbf{\Omega}\sqrt{\gamma}\,. \tag{5.35}$$

The first term is invariant under the rotation SO(2) which rotates ϕ into χ. Using the fact that $\nabla\cdot\mathbf{\Omega}=0$, and an integration by parts, the last term may be re-written in the manifestly SO(2)-symmetric form

$$\frac{1}{2}\left(\phi\nabla\chi-\chi\nabla\phi\right)\cdot\mathbf{\Omega}\,. \tag{5.36}$$

So even in the curved background we see that the dimensional reduction results in a duality-symmetric lagrangian. This is also the case for our new model of section 4.3.3.

Note that if we define $\mathbf{H}=-\nabla\chi$ the square root is the same as in flat space except that the inner product $\cdot$ is taken in the metric γ_{ij}.

To obtain the (anti-)self-dual solutions one needs to consider t, ω and ϕ to be pure imaginary. One sets

$$\phi=\pm i\chi\,. \tag{5.37}$$

In flat space one may also include in the Dirac-Born-Infeld action a scalar field y representing the transverse displacement of the 3-brane, In the stationary case one finds that the SO(2) duality symmetry is enhanced to an $\mathrm{SO}(2,1)$ [22]. This $\mathrm{SO}(2,1)$ symmetry contains an $\mathrm{SO}(1,1)$ subgroup allowing ϕ to be boosted into y. The fixed points , i.e. the solutions with $\phi=\pm y$ are BPS and correspond to BIon solutions representing strings attached to the 3-brane. We have included this and repeated the previous steps. The resultant dualized lagrangian is written as

$$\sqrt{\gamma}e^{-2U}\left(1-\sqrt{\det\left(\mathbf{v}_R\cdot\mathbf{v}_S-\mathcal{G}_{RS}\right)}\right)-\chi\nabla\phi\cdot\Omega\sqrt{\gamma}\,, \tag{5.38}$$

where $R,S=1,2,3$ and the vectors $\mathbf{v}_R$ are defined as

$$\mathbf{v}_1\equiv\nabla\phi\,,\qquad\mathbf{v}_2\equiv\nabla\chi\,,\qquad\mathbf{v}_3\equiv e^{U}\nabla y\,, \tag{5.39}$$

the metric in the internal space is defined as $\mathcal{G} = \mathrm{diag}(-1,-1,1)$. The first term of this action looks invariant under the rotation in this internal space and seems to have symmetry SO(2,1). However, we find that the presence of e^U in the definition of $\mathbf{v}_3$ prohibits us to write the symmetry in terms of scalar variables, and breaks the SO(2,1) invariance down to the SO(2) duality subgroup. In addition, the second term is invariant under only the SO(2) group.

One may also couple to gravity. One adds to (5.32)

$$\frac{1}{4}\sqrt{\gamma}\left(R(\gamma) - 2|\nabla U|^2 + \frac{1}{2}e^{4U}|\mathbf{\Omega}|^2\right), \tag{5.40}$$

where we are using a notation $4\pi G = 1$. In order to maintain the constraint $\nabla\cdot\mathbf{\Omega} = 0$, one also adds

$$\frac{1}{2}\sqrt{\gamma}\psi\nabla\cdot\mathbf{\Omega}\,, \tag{5.41}$$

so that there are now two Lagrange multiplier fields ψ and χ. The quantity ψ is usually called the NUT-potential. In the absence of matter, elimination leads to the $\mathrm{Sp}(2,\mathbb{R})$ invariant action

$$\sqrt{\gamma}\left(\frac{1}{4}R(\gamma) - \frac{1}{2}|\nabla U|^2 - \frac{1}{2}e^{-4U}|\nabla\psi|^2\right). \tag{5.42}$$

In the case of riemannian, rather than lorentzian metrics one considers ψ to be pure imaginary and the self-dual metrics with triholomorphic Killing vectors are obtained by setting $e^{2U} = \pm i\psi$.

If one couples Einstein gravity to Maxwell theory and proceeds with the elimination of the two Lagrange multipliers, we obtain

$$\begin{aligned}\sqrt{\gamma}\bigg[\frac{1}{4}R(\gamma) &- \frac{1}{2}|\nabla U|^2 - \frac{1}{2}e^{-4U}|\nabla\psi - \phi\nabla\chi + \chi\nabla\phi|^2 + \\ &+ \frac{1}{2}e^{-2U}\left(|\nabla\phi|^2 + |\nabla\chi|^2\right)\bigg]. \end{aligned}\tag{5.43}$$

It is well known that the duality group of Maxwell theory SO(2) is promoted to an SU(2,1) symmetry which includes SO(2). The group SU(2,1) is what Hull and Townsend [24] call the U-duality group in this case.

If we repeat this procedure for Born-Infeld one finds the lagrangian

$$\begin{aligned}\sqrt{\gamma}\bigg[\frac{1}{4}R(\gamma) &- \frac{1}{2}|\nabla U|^2 - \frac{1}{2}e^{-4U}|\nabla\psi - \phi\nabla\chi + \chi\nabla\phi|^2 - \\ &- e^{-2U}\sqrt{(1-|\nabla\phi|^2)(1-|\nabla\chi|^2) - (\nabla\phi\cdot\nabla\chi)^2 + e^{2U}}\bigg]. \end{aligned}\tag{5.44}$$

We observe that the symmetry SU(2,1) is not present. It seems therefore that if there is some extension of Born-Infeld containing gravity which is invariant under the full SU(2,1) U-duality group, it is not simply the obvious coupling of Einstein gravity coupled to Born-Infeld theory just described.

6. Examples of solutions in curved space

If we do not consider the existence of the b-field, then our procedure of sections 3 and 4 shows that many examples of self-dual Maxwell fields on curved manifolds studied in previous literature can survive even when they are generalized to duality invariant non-linear electrodynamics. Thus in this section we concentrate on the situation where the b-field is turned on.

6.1 Born-Infeld instantons on curved manifolds with b-field

One motivation for this paper was to consider non-linear electrodynamics in a constant or almost constant background b-field in curved riemannian 4-manifold $\mathcal{M}$, as for example in the work of ref. [25] where a single Born-Infeld field was considered. From our work above, if the theory satisfies the three requirements (4.8)–(4.10), to get a solution, we may take $\mathcal{F}$ to be a (anti-)self-dual Maxwell field. Thus from now on we shall take Born-Infeld instanton to mean an L^2 Maxwell field F of fixed duality together with a b-field of the same duality which is closed and covariantly constant or at least covariantly constant near infinity.

Another possible motivation is to see to what extent the S-duality results for Maxwell theory obtained in ref. [26] continue to hold in Born-Infeld theory.

Maxwell theory on riemannian four-manifolds has been much studied. In particular a fair amount is known about L^2 harmonic forms though many mathematical problems remain. In what follows we shall focus on the problem of finding suitable b-fields, quoting the results about L^2 harmonic forms that we need.

6.1.1 Closed manifolds

If the manifold $\mathcal{M}$ is closed, that is compact without boundary, $\partial\mathcal{M} = 0$, and the solution is square integrable then we are led to standard Hodge-de-Rham theory. The number of linearly independent (anti-)self-dual Maxwell fields is topological and is given by the Betti-numbers $b_2^{\pm}(\mathcal{M})$. Note that even if $\mathcal{M}$ is symplectic, that is even if admits a closed non-degenerate 2-form ω such that $d\omega = 0$, and we regard the manifold $\mathcal{M}$ as non commutative in the sense that we replace the usual commutative product on smooth functions by a deformation of the Poisson algebra [27], the *topology* of $\mathcal{M}$ will not change. If $\mathcal{M}$ is taken to be Kähler then the Kähler form will be of fixed duality. We shall adopt the convention that it is self-dual. Because the manifold is Kähler it will also be covariantly constant.

Thus in the Kähler case it is natural to take, as suggested in ref. [25] b to be proportional to the Kähler form ω. The Born-Infeld instantons would then be the harmonic forms with the same duality which are not covariantly constant. A special case arises when $\mathcal{M}$ is Hyper-Kähler. In that case, regardless of whether $\mathcal{M}$ is closed or compact we have a three covariantly constant 2-forms all of which may be taken to be self-dual. The only non-flat simply connected example is $K3$ in which case the non-constant two forms are all anti-self-dual and form a 19-dimensional family. Thus there are no Born-Infeld instantons on $K3$.

6.1.2 Hyper-Kähler ALE and ALF spaces

This is closer to the situation envisaged in ref. [25]. In general a compact or non-compact hyper-Kähler manifold admits no L^2 harmonic forms with the same duality as the Kähler forms. Thus again we do not expect any Born-Infeld instantons in this case.

In particular in ref. [25] a blow up of $\mathbb{R}^4/\mathbb{Z}_2$ was considered. If the manifold were Hyper-Kähler, it would be the Eguchi-Hanson metric which is one of the family of metrics admitting a triholomorphic Killing vector $\partial/\partial\tau$. We have

$$ds^2 = V^{-1}(d\tau + \omega \cdot d\mathbf{x})^2 + V(d\mathbf{x})^2\,, \tag{6.1}$$

with $\operatorname{curl}\omega = \nabla V$. We take

$$V = \epsilon + \sum \frac{1}{|\mathbf{x} - \mathbf{x}_i|}\,. \tag{6.2}$$

If $\epsilon = 0$ the metric is ALE. If $\epsilon = 1$ it is ALF. To get the Eguchi-Hanson metric we set $\epsilon = 0$ and take two centres. The metric is believed to admit just one L^2 Maxwell field which is anti-self dual. It is associated with the non-trivial topology and it is easy to write it down explicitly. Strictly speaking there seems to be no completely mathematically rigorous proof, in the manner of ref. [28] that this 2-form exhausts the L^2 cohomology of Eguchi-Hanson but this conjecture is supported by index theorem calculations [29]. If $\epsilon = 0$ and for k centres there is believed to be just a $(k-1)$-dimensional space of anti-self-dual L^2 harmonic two-forms associated with the topology. The ALF case is very similar, except in addition to the anti-self-dual two-forms expected from the topology, the exact anti-dual two form

$$d\left(V^{-1}(d\tau + \omega \cdot d\mathbf{x})\right) \tag{6.3}$$

is also in L^2. In the case of one centre, i.e. for the Taub-NUT solution it is known [28] that this is the only L^2 harmonic two-form. Thus again we get no Born-Infeld instantons in this case.

6.2 Almost Kähler manifolds

Rather than taking b to be covariantly constant as it is for a Kähler manifold, one could also consider an almost Kähler manifold. This has an almost complex structure J such that $J^2 = -1$ with respect to which the metric is hermitean, in other words for any two vectors X and Y

$$g(JX, JY) = g(X, Y)\,. \tag{6.4}$$

The associated two form

$$\omega(X,Y) = g(X,JY) \tag{6.5}$$

is closed but not covariantly constant. However, just as for a Kähler manifold, the magnitude squared, $\frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta}$, of the two form ω is constant. The difference from a Kähler manifold is that the almost complex structure J is no longer integrable.

A theorem of Sekigawa [30] tells us that if $\mathcal{M}$ is a compact Einstein manifold with non-negative scalar curvature then any almost Kähler manifold must in fact be Kähler. Perhaps for that reason, almost-Kähler metrics have not been much studied by physicists. However, if $\mathcal{M}$ is not compact, then there may exist complete Ricci-flat almost Kähler manifolds which are not Kähler. One particularly intriguing example, and in fact the only one known to us, is of the form (6.1) [31]. It is however incomplete. The metric and some of its properties are described in detail in ref. [32]. In ref. [33, 34] the uniqueness of this example is discussed.

The metric is obtained by taking the harmonic function V to be proportional to one of the spatial coordinates z say. Thus it is given by

$$ds^2 = z^{-1}(d\tau + xdy)^2 + z\left(dx^2 + dy^2 + dz^2\right). \tag{6.6}$$

Clearly there is a singularity at $z = 0$ but the metric is complete as $z \to \infty$. As explained in ref. [32] the singularity may be removed by regarding the metric (6.6) as the asymptotic form of a complete non-singular hyper-Kähler metric on the complement of a smooth cubic curve in CP^2.

The metric (6.6) has a Kaluza-Klein interpretation (reducing on the coordinate τ) as a magnetic field in the z-direction which is uniform in x and y. Appropriately enough the isometry group is of Bianchi type II, or Heisenberg group. The Maurer-Cartan forms are $(dx, dy, d\tau + xdy)$. It is well known that the Heisenberg group is the symmetry group of a particle moving in a uniform magnetic field and in some circumstances the spatial coordinates x and y may be regarded as non commuting. Thus it is a very tempting to speculate that this metric and its smooth resolution may have something to do with curved non-commutative spaces. In any event it encourages its further investigation.

One easily checks that the three (self-dual) Kähler forms are

$$dx \wedge (d\tau + xdy) + zdz \wedge dy\,, \tag{6.7}$$

$$dy \wedge (d\tau + xdy) + zdy \wedge dz\,, \tag{6.8}$$

and

$$dz \wedge (d\tau + xdy) + zdx \wedge dy\,. \tag{6.9}$$

In addition there is a circle of anti-self-dual almost Kähler forms

$$\cos\theta\Big(dx \wedge (d\tau + xdy) - zdx \wedge dy\Big) + \sin\theta\Big(dy \wedge (d\tau + xdy) - zdy \wedge dy\Big)\,, \tag{6.10}$$

where θ is a constant. The triholomorphic Killing field $\partial/\partial\tau$ gives rise to an anti-self dual Maxwell field of the form

$$d\left(z^{-1}(d\tau + xdy)\right) \tag{6.11}$$

whose magnitude squared is proportional to z^{-4}. Since the volume element is $zd\tau dxdydz$, the L^2 norm converges as $z \to \infty$ but diverges as one approaches the singularity at $z = 0$.

An interesting question is whether the almost Kähler structure (6.10) or the Maxwell field (6.11) extend smoothly over the resolution. If the almost Kähler structure extends, it would give a complete Ricci-flat example of an almost Kähler structure which is not Kähler.

6.3 A black hole in an external b-field

In the previous subsection, we showed that ALE and ALF spaces in a background b-field do not admit interesting Born-Infeld instantons. The situation changes if we consider euclidean Schwarzshild solutions. The metric is

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \,. \tag{6.12}$$

The L^2 harmonic two-forms are spanned by [35]

$$F_m = \sin\theta \, d\theta \wedge d\varphi \tag{6.13}$$

and

$$F_e = *F_m = \frac{(d\tau \wedge dr)}{r^2} \,. \tag{6.14}$$

Thus $F_m \pm F_e \equiv F_\pm$ is a (anti-)self-dual Maxwell instanton. To get a b-field, consider a harmonic two-form B

$$B = d\left(\frac{r^2}{2}\sin^2\theta d\varphi\right) . \tag{6.15}$$

The Maxwell field B is neither (anti-)self-dual nor square-integrable. It looks like a uniform magnetic field near infinity. However,

$$b_\pm = B \pm *B \tag{6.16}$$

is (anti-)self-dual and is approximately covariantly constant near infinity. Thus in general (anti-)self-dual solution of Maxwell equations in a (anti-)self-dual background b-field may be obtained by taking an arbitrary linear combination of $F_\pm$ and $b_\pm$. It is amusing to speculate that this solution may have something to do with "non-commutative black holes".

A smooth square integrable solution of the general non-linear electrodynamic equations of motion which is not self-dual may be obtained by taking $P = F_e$. This satisfies the equation of motion $d * P = 0$. Now one may calculate F using the constitutive relations. This will give something of the form $F = f(r) d\tau \wedge dr$, for a function $f(r)$ which will depend upon the particular theory. One has $dF = 0$ and therefore it also satisfies the Bianchi identities.

6.4 Kaluza-Klein monopoles in constant b-fields

A similar construction to that of the previous section may be applied to axisymmetric Hyper-Kähler metrics. Let us suppose that the metric is of the form (6.1) but where V and ω are independent of the angular coordinate $\varphi = \arctan(\frac{y}{x})$. We may choose a gauge in which

$$ds^2 = V^{-1}(d\tau + \nu d\varphi)^2 + V\left(dz^2 + d\rho^2 + \rho d\varphi^2\right), \tag{6.17}$$

where $\rho^2 = x^2 + y^2$ and V and ν depend only on ρ and z.

The Killing vector $m^\alpha \partial_\alpha = \partial/\partial\varphi$ is not triholomorphic, and the two-form

$$d\left(g_{\alpha\beta} m^\alpha dx^\beta\right) \tag{6.18}$$

is an anti-self-dual solution of Maxwell's equations. If the metric is ALF then near infinity (6.18) will look like a magnetic field and be almost constant. Thus we could take it as our b-field. To get Born-Infeld instantons we can now take anti-self-dual L^2 harmonic two-forms. The simplest case would be the Taub-NUT metric. It admits a unique [28] L^2 Maxwell field which is anti-self-dual and it will persist as one turns on the b-field.

7. Conclusion and discussion

In this paper, we have analyzed non-linear electrodynamics in curved space especially from the viewpoint of self-duality and self-dual configurations. We have observed that various properties of the self-dual configurations can be derived from the duality invariance of the system. If the equations of motion are duality invariant, then the self-dual configurations satisfy the following properties: (i) they also satisfy the natural duality $F = \pm * P$, (ii) they are also the solutions of equations of motion, (iii) their stress tensors vanish.

In the U(1) case these three properties are equivalent with a single constraint on the form of the lagrangian. However, if we consider several gauge fields, then the situation is rather different. We analyzed the form of the constraint and gave a general argument. We presented a new action which possesses $\mathrm{SO}(n) \times \mathrm{SO}(2)$ duality symmetry and the above three properties. The extension to include axion and dilaton gives an enhanced duality group, and this was checked for our new action. The relation to Legendre duality was clarified.

JHEP09(2000)013

Motivated by the work [25], we studied the inclusion of self-dual b-field background. We gave some explicit examples of these "non-commutative manifolds" and studied self-dual configurations of the gauge fields on them.

In the case that the curved background admits a Killing vector field, we obtained the dimensional reduction of the Born-Infeld action and it's coupling to Einstein gravity. Interestingly, while the SO(2) duality symmetry becomes manifest, other symmetries such as the SO(2, 1) symmetry which appears in flat space when one introduces a transverse scalar or the SU(2, 1) U-duality group which emerges when one couples Einstein gravity to Maxwell theory are broken. This may indicate that there are other more symmetrical ways of coupling Born-Infeld theory and gravity consistent with U-duality.

In this paper we have seen various dualities of non-linear electrodynamics. We can summarize them in the following table:

(1) Legendre (discrete) self-duality	:	$F \leftrightarrow *P$
(2) SO(2) duality invariance	:	Rotation among $(F,\, *P)$
(3) $\mathrm{Sp}(2,\mathbb{Z})$ invariance	:	Discrete rotation among $(F,\, *P,\, \Phi,\, a)$
(4) $\mathrm{Sp}(2,\mathbb{R})$ invariance	:	Continuous version of (3)

We have a flowchart representing the relations between the above dualities as

$$(4) \Longrightarrow (3) \Longrightarrow (1)\,, \qquad (4) \Longrightarrow (2) \Longrightarrow (1).$$

Some comments concerning the results of this paper are in order.

In section 4, we assumed that lagrangian is written only in terms of X_i and Y_i. However, in general the combination $F^{(i)}_{\mu\nu}F^{(j)\mu\nu}$ with $i \neq j$ can appear. Inclusion of this term gives rise to a problem : we cannot write the energy-momentum tensor in terms of these Lorentz invariant quantities. Since the general duality condition on the matrix B (4.22) involves this cross term in an essential way, a generalization of our argument must be provided in order to see the full content of the structure of the duality groups.

Our original motivation was to see what sort of lagrangian is consistent with self-dual configurations in a curved background. This may some insight into determining the form of the D-brane action. Note that even in a flat background the precise form of the non-abelian D-brane action is not yet known with certainty. Integrating out the off-diagonal gauge field components will produce a complicated form, but it must be constrained by our duality condition in the case of many U(1) gauge fields.

In ref. [36], an abelian Born-Infeld generalization of the topological Donaldson-Witten model was constructed, and it was shown that BRST-invariant quantities coincide with those of the Maxwell-Donaldson-Witten topological model. As stated in ref. [36], the explicit form of the lagrangian was not used, merely the properties of it's extrema. It seems likely therefore that topological theories constructed from other non-linear electrodynamical theories satisfying our conditions will also have the same BRST-invariant quantities as Maxwell-Donaldson-Witten theory.

We would like to comment on the relation to the recent work on so-called non-linear BPS equations [6], [37]–[41]. The configurations considered in this paper are solutions of linear BPS equations, that is self-dual equations. But in general under the presence of a b-field, the duality condition may be relaxed because of the freedom introduced by the boundary condition of the gauge field at spatial infinity. In most previous work only flat backgrounds were adopted. In this paper we have shown that the linear BPS equation survives when we go to a curved background, so it is expected that the non-linear BPS equation can be derived in a curved background. One supporting reason is that the solutions of the non-linear BPS equations are simply derived from the linear BPS solution by making use of a target space rotation [39, 40, 42]. We leave these issues to the future work.

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